Geometrical Renormalization Groups: Perfect Deconstruction Actions

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Abstract

By combining two distinct renormalization group transformations, opposing scale transformations, we obtain a composite transformation which does not rescale the system, and drives it to a "geometrical" fixed point, controlling the effective geometry and locality. The latticized (deconstructed) action for an extra-dimensional field theory becomes a "perfect action," with a linear ladder spectrum for N nodes.

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1 Introduction

When theories of extra dimensions are latticized [1] or "deconstructed" [2], we obtain gauge invariant effective Lagrangians for KK-modes in $1 + 3$ dimensions. The structure of the compactified space is mapped into the structure of the latticized matter theory. The KK spectrum of the lattice action is not linear with only nearest neighbor hopping terms, but takes the form of a phonon spectrum, *e.g.*, $m_n \sim \Lambda \sin(n\pi/N)$ for the modes on a latticized compactified S_1. The spectrum is only approximately linear for small n, and the nearest neighbor latticized bulk is not a true effective Lagrangian for N KK modes. Incorporating sufficiently many operators into $\mathcal{L}_{eff}$ with only N degrees of freedom, we can always linearize the spectrum.

There is, however, a deeper issue here. Deconstruction replaces the extra dimension by a matter theory living in $3 + 1$ dimensions, and the notion of the extra dimension's geometry is lost. What, therefore, is the symmetry principle which governs the true $\mathcal{L}_{eff}$ for N degrees of freedom with a linear ladder KK mode spectrum, intrinsic to the deconstructed theory and making no reference to extra dimensions? This issue is surely related to the translational invariance in the continuum extra dimensional theory. However, if we write down an arbitrary continuum extra dimensional theory, we will have the linear spectrum only if the theory has canonical quadratic kinetic terms. Higher dimension operators, containing derivatives, must be absent or suppressed. The linearity of the KK-mode spectrum has as much to do with *locality* as with translational invariance. We may therefore ask if there is a more general statement about a theory which contains both the locality and geometry.

The set of all possible matter theories that contain the degrees of freedom of a given latticized extra dimension, but which also contain more arbitrary interactions, hopping terms, etc., has been dubbed "theory space" [2]. For our present discussion we want the number of degrees of freedom, N, held fixed, as a kind of "microcanonical ensemble" of theories. We will argue that there is a particular operator, Q, which acts upon the Lagrangians of the theory space, mapping the theory space into itself. Q will be built out of the product of pairs of renormalization group (RG) transformations. One of these RG transformations integrates in more degrees of freedom ("decorates" the theory) while the other RG transformation thins the degrees of freedom by a block-spin renormalization (Bell-Wilson transformation [3]).

This contrasts typical RG transformations which rescale a theory while thinning de-

grees of freedom, carrying a local "reference scale of physics" μ into a new scale $\lambda\mu$. Q is a product of an "up scaling" times a "down scaling" transformation, $i.e.$, $\lambda \times \lambda^{-1}$, and is net scale invariant. Q is suited to the situation of compactification of extra dimensions where there are two fixed physical scales, the compactification radius R, and the fundamental, or "string" scale, Λ (which may be viewed as the perturbative unitarity bound for longitudinal KK-mode scattering in the deconstructed theory [1]).

The class of all Q's and their fixed points in theory spaces is no doubt very large and rich. Recently we constructed a particular intriguing and nontrivial example, a *fractal field theory of extra dimensions* [4]. The fractal theory is constructed on a lattice with coordination number s via a recursive procedure in which $\sim s^{k+1}$ lattice sites are created at kth order. The Q operation is defined by a pair of scaling operations which map $k \to k - 1$, but preserves scale, involving RG concepts borrowed from the Ising model [5, 6, 7]. Q becomes a symmetry operation in the "continuum limit" $k \to \infty$, and one discovers a fixed point theory with a "critical exponent," ϵ, characterizing the energy distribution of KK modes, $\mathcal{N}(E) \sim (E/M)^\epsilon$. Here M is a nontrivial RG invariant scale, the analogue of Λ_{QCD}, and corresponds to the mass of the first KK-mode, i.e. M^{-1} acts like the effective compactification scale. On scales above M the theory effectively exists in $D + \epsilon$ dimensions, where D is integer, and ϵ is an irrational number. The fractal theory is a nontrivial, candidate for a *finite quantum field theory* to all orders of perturbation theory. The finiteness traces to the RG transformation, implying that pointlike vertices are meaningless, since they are replaced by simplices under the transformation Q (analogous to string theory having undefined point-like vertices). Hence, Q invariance can define a nontrivial short-distance fractal theory with large distance Lorentzian geometry, In this example Q invariance is more fundamental in defining the theory than is geometry, which becomes meaningless at short distances. Perhaps the physical world does not respect the notion of geometry at very short distances, but, rather, is defined by recursion.

Thus, there exists a special point in theory space, representing a Lagrangian $\mathcal{L}^*$, which is a fixed point under Q, i.e., $Q(\mathcal{L}^*) = \mathcal{L}^*$. Q provides an effectively local geometric theory, but makes no reference to the existence of an extra dimension. Q is defined only as a procedure that acts on theory space.

Let us first consider in general the relationship of our problem, the construction of Q, for latticized extra dimension, to the corresponding problem in an approximately scale invariant theory, $e.g.$, short distance lattice QCD.

2 Perfect Actions

The idea of constructing RG tranformations and finding their fixed points to improve lattice actions lies in the arena of "perfect actions" from lattice QCD [3, 8, 9]. One seeks improved lattice actions which, for QCD, are better descriptions of the physics in the limit in which the dynamics is approximately scale invariant, *i.e.*, at very short distances where the stress tensor trace anomaly, is approaching zero, $T_\mu^\mu \propto \beta(g)/g \to 0$. The concept of perfect actions begins, in particular, with the block-spin renormalization group transformation of Bell and Wilson [3], and developed subsequently in QCD lattice gauge theory, by Wiese [8], Niedermayer, Hasenfratz, *et al.* , [9]. Perfect actions thus respect, in principle, a well defined symmetry operation: they are invariant, or fixed points, under block-spin renormalization group transformations that are scale transformations. We say "in principle" because the implementation of this idea is often only approximate, corrected perturbatively, where possible. We evidently seek an analogue of the perfect action for a latticized compactified bulk, and the principle which determines it.

A continuum free theory is trivially scale invariant under Bell-Wilson BSRG transformations when a certain parameter of the transformation, λ, is chosen to drive the theory toward the Gaussian Fixed point. With a less trivial dispersion relation, $p_0^2 = \vec{p}^2 + \vec{p}^4/M^2 + ...$ the theory flows into the infrared free field theory $p_0^2 = \vec{p}^2$ under successive applications of the Bell-Wilson transformation for $\lambda^* = 1/\sqrt{2}$ (Bell and Wilson introduce the parameter b which corresponds to our λ up to a dimension dependent factor; presently $d = 1$ in their language, and the critical value of $\lambda = \lambda^*$ matches b^*). Perfect actions for QCD are in principle the fixed point actions under this transformation. They are, in principle, implemented in the fully interacting quantum field theory. They are typically quasilocal, involving suppressed (next-to)^n nearest neighbor interactions. Approximate forms have been constructed, [9], and the Gaussian perfect action gives the best starting point for a perturbative quantum perfect action.

In the case of compactified extra dimensions and their deconstructed description, the situation with respect to scale invariance is drastically different than that of QCD. Here we *do not* have scale invariance. We have instead two relevant scales: (1) a high energy or fundamental, or "string scale," Λ, and (2) the low energy compactification scale $M = 1/R$. The compactification scale $\sim 1/R$ is identified with the mass of the lowest KK mode. RG transformations of interest must therefore act within the range of scales $\Lambda >> \mu >> M$, but must not affect Λ and M, which define the theory. Indeed, a continuum extra
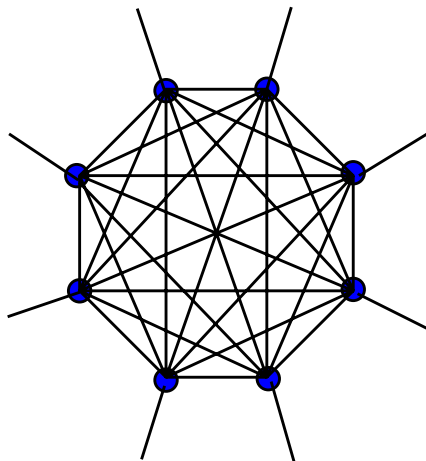
Figure 1: A "complete octagon" has each of the 8 sites linked to every other site. The eigenmode spectrum is one zero mode and 7 modes of mass $\sim \sqrt{8}\Lambda$.

dimensional theory is generally never scale invariant. The stress tensor trace is nonzero in $D \neq 4$, and the coupling constant is dimensional. This is the origin of the power-law running of the coupling constant. In deconstruction, the corresponding $D = 4$ stress tensor trace matches the $D \neq 4$ stress tensor trace through the presence of the "linking Higgs" fields, which have nonzero VEV's of order Λ explicitly breaking scale invariance.

In deconstruction, the continuous translational symmetry in an extra dimension is replaced by a discrete Z_N symmetry. The kinetic terms become Z_N invariant hopping terms. Locality implies that hopping terms do not have strong linking to distant sites. This is the key to having anything resembling a ladder spectrum. For example, if one constructs a "complete" lattice of N sites, linking each site to every other site with a hoping term strength Λ, as in Figure (1), the spectrum will be have a single zero mode, and $N - 1$ degenerate levels of mass $\sim \sqrt{N}\Lambda$. This is as far from a ladder spectrum as one can get.

It is locality, therefore, together with Z_N invariance, that sets up the hierarchy between the high scale Λ and the compactification scale $1/R$, and uniformly populates the spectrum on scales $\Lambda > \mu > 1/R$ with KK modes. Nevertheless, any restriction to local, mainly nearest neighbor, links appears to be arbitrary from a "theory space" point of view. The Q symmetry however, will select the fixed point theory, enforcing quasi-locality and a linear ladder spectrum.

We can understand Q with the following metaphor. Suppose we are climbing a tall cell-phone tower ladder, and we are in the middle of the ladder at a scale μ, roughly midway between the top of the ladder (Λ) and bottom (M). When we are midway up the ladder

($\Lambda >> \mu >> M$), we are far from any fundamental scale, and the effective Lagrangian should be insensitive to changes in μ. The ladder reflects an approximate symmetry in momentum space in that, "hopping up" several rungs, the ladder should appear to us to be the same physical environment. Suppose we double Λ/M, thus doubling the height of the ladder while holding the rung spacing on the ladder fixed. This maps our ladder with N rungs into one with $2N$ rungs. Now we follow this with another operation that integrates out $1/2$ of the rungs and rescales the ladder height by $1/2$. Under these combined operations we have no net rescaling, but trivially recover the original linear ladder with N rungs.

More generally, however, if the ladder is not uniform, we might double the number of rungs by some procedure, k times, thus increasing its height by 2^k. Then we "integrate out" half the rungs k times by averaging over the locations of rungs that are integrated, suitably smearing out the local inhomogeneities. We expect then that an arbitrary inhomogeneous ladder becomes flat in the large k limit. Thus, while the spectrum of the nearest neighbor theory is not flat, having the phonon $m_n \sim \Lambda \sin(n\pi/N)$ structure, under application of this kth order RG transformation we expect it to flow toward a fixed point, until it flattens, $m_n \sim \Lambda n\pi/N$. An action which is a fixed point under this transformation should be the perfect action we seek.

In Section 3 we will make this more precise in the language of "decoration" and "dedecoration" transformations, which are defined in configuration space. We first define the product of a decoration transformation D and a dedecoration transformation D^{-1} to be the identity transformation. We illustrate our general technique by carrying out an analysis of the trivial case of k decorations followed by k dedecorations, $(D^{-1})^k D^k$ to set up the subsequent nontrivial case of interest.

We then consider decoration transformations followed by Bell-Wilson (BW) transformations [3] acting on scalar field theories. The BW transformation acts in momentum space to define the block spin variables, and introduces a new parameter, λ, which is the "wave-function renormalization constant" of the block-spin variable. Thus, we perform a decoration transformation D^k followed by a Bell-Wilson transformation k times, B^k, which groups the original sites into "block spins" and maps the theory back to the original N branes. We thus have Q defined through $Q^k = B^k D^k$ and we are interested in the formal limit as $k \to \infty$.

The Gaussian fixed point of the BW transformation is found for the special value $\lambda = \lambda^* = 1/\sqrt{2}$. The spectrum, under the Q transformation for large k flows toward the
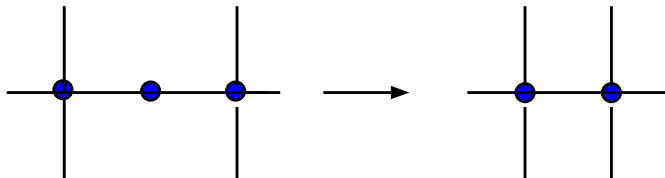
Figure 2: The 3-chain $\to$ 2-chain dedecoration transformation integrates out the internal field and renormalizes the endpoint fields' kinetic terms and mass terms.

linear ladder spectrum. Indeed, the spectrum has the form $m_n^{(k)} \to 2^{k+1}\Lambda\sin(2^{-k}n\pi/N)$ under $B^k D^k$, and flows toward the flat $m_n^{(k)} \to 2\Lambda n\pi/N$ spectrum as $k \to \infty$. The action of the deconstructed theory which is itself a fixed point under $B^k D^k$ as $k \to \infty$ corresponds to "a perfect action," analogous to perfect actions in lattice QCD. This codifes the sense in which the ladder spectrum is special, within the context of theory space.

3 Transformations for Deconstructed Lattice Scalars in $1+4$

We begin by considering transformations which augment or thin the degrees of freedom of $1+3$ theories of many complex scalar fields. These transformations stem from symmetries noticed long ago in the Ising model, [5, 6, 7]. In the language of Ising models a single spin$_1$-link-spin$_2$ combination in the Hamiltonian can always be "decorated," *i.e.*, written as a spin$_1$-link-spin$'$-link-spin$_2$ interaction. That is, we can "integrate in" the new spin$'$, or "decorate" the original single link. Thus, an N-spin system can be viewed as a $2N$ spin system upon decorating. The decorations can be arbitrarily complicated, involving many new spins. Conversely, we can "integrate out" or "dedecorate" the spins internal to a chain whose endpoint spins are then renormalized (Fig.(2)).

Decoration is an exact transformation for Ising spins, and continuous spins (e.g., "spherical models" are spin systems which correspond to our models in the absence of kinetic terms). Presently our "spins" are fields that have $1+3$ kinetic terms. For us decoration and dedecoration transformations are exact transformations only in the limit of

very large cut-off Λ. This happens because we perform decoration transformations truncating on quartic derivatives, such as $|\partial^2 \phi_a|^2 / \Lambda^2$. This is, nonetheless, a good approximate transformation in the $\Lambda \to \infty$ limit, or for the low lying states in the spectrum. These transformations become symmetries in the continuum limit when the theory is classically scale invariant, $i.e.$, $\mu^2 = 0$ and $\Lambda \to \infty$. The $1+3$ kinetic terms undergo renormalizations under these transformations, and thus distinguish the present construction from that of a spin model.

3.1 Decoration Transformations

Consider an N complex scalar field Lagrangian in $1 + 3$, which can be viewed as a deconstructed S_1 compactified extra dimension with periodic boundary conditions:

$$\mathcal{L}_N = Z_N \sum_{a=1}^{N} |\partial \phi_a|^2 - \Lambda_N^2 \sum_{a=1}^{N} |\phi_a - \phi_{a+1}|^2 - \mu_N^2 \sum_{a=1}^{N} |\phi_a|^2 \tag{3.1}$$

and assume periodicity, hence $\phi_{N+a} = \phi_a$. It is convenient to allow for noncanonical normalization of the kinetic terms, and we thus display the arbitrary wave-function renormalization constant Z_N.

A decoration transformation replaces $\mathcal{L}_N$ by a new Lagrangian $\mathcal{L}_{2N}$ with $2N$ fields and new parameters.

$$\mathcal{L}_N \to \mathcal{L}_{2N} = Z_{2N} \sum_{a=1}^{2N} |\partial \phi_a|^2 - \Lambda_{2N}^2 \sum_{a=1}^{2N} |\phi_a - \phi_{a+1}|^2 - \mu_{2N}^2 \sum_{a=1}^{2N} |\phi_a|^2 \tag{3.2}$$

and again assume periodicity, hence $\phi_{2N+a} = \phi_a$ The new parameters are chosen so that upon integrating out every other field (e.g., those with even a in the sums), we recover the original Lagrangian $\mathcal{L}_N$.

We define the new decorated Lagrangian parameters by demanding that $\mathcal{L}_{2N}$ be equivalent to $\mathcal{L}_N$ upon "dedecorating," $i.e.$, integrating out every other field. We seek the relation between the parameters X_{2N} of $\mathcal{L}_{2N}$, and the X_N of $\mathcal{L}_N$. It is useful to consider $\mathcal{L}_{2N}$ as a sum over 3-chains:

$$\mathcal{L}_{2N} = \sum_{\substack{n=1 \\ n \, odd}}^{2N-1} \mathcal{L}_{n,n+2} \tag{3.3}$$

Each 3-chain involves three fields. The first 3-chain is:

$$\mathcal{L}_{1,3} = \frac{1}{2} Z_{2N} (|\partial \phi_1|^2 + 2|\partial \phi_2|^2 + |\partial \phi_3|^2) - \Lambda_{2N}^2 |\phi_1 - \phi_2|^2 - \Lambda_{2N}^2 |\phi_2 - \phi_3|^2 - \frac{1}{2} \mu_{2N}^2 (|\phi_1|^2 + 2|\phi_2|^2 + |\phi_3|^2) \tag{3.4}$$

The fields ϕ_1 and ϕ_3 share half their kinetic terms and μ^2 terms with the adjacent chains, thus carry the normalization factors of $1/2$ within the chain. ϕ_2 can be thus viewed as a "decoration" of the chain. We integrate out the internal field ϕ_2 and obtain an equivalent renormalized chain. Integrating out ϕ_2:

$$\mathcal{L}_{1,3} = \frac{1}{2}Z_{2N}(|\partial\phi_1|^2 + |\partial\phi_3|^2) - (\Lambda_{2N}^2 + \frac{1}{2}\mu_{2N}^2)(|\phi_1|^2 + |\phi_3|^2) +$$
$$+ \Lambda_{2N}^4(\phi_1 + \phi_3)^\dagger \frac{1}{Z_{2N}\partial^2 + 2\Lambda_{2N}^2 + \mu_{2N}^2}(\phi_1 + \phi_3) \tag{3.5}$$

Expanding in the derivatives to $\mathcal{O}(\partial^2)$, integrating by parts, regrouping terms, and relabeling parameters gives:

$$\mathcal{L}_{1,3} = \frac{1}{2}Z_N(|\partial\phi_1|^2 + |\partial\phi_3|^2) - \Lambda_N^2|\phi_1 - \phi_3|^2 - \frac{1}{2}\mu_N^2(|\phi_1|^2 + |\phi_3|^2) - \delta_N|\partial(\phi_1 - \phi_3)|^2 + \mathcal{O}(\partial^4/\Lambda^2) \tag{3.6}$$

where we now have the relationship between the parameters of eq.(3.5) and eq.(3.6):

$$Z_N = Z_{2N}\frac{8\Lambda_{2N}^4 + 4\Lambda_{2N}^2\mu_{2N}^2 + \mu_{2N}^4}{4\Lambda_{2N}^4 + 4\Lambda_{2N}^2\mu^2 + \mu_{2N}^4} \approx 2Z_{2N}$$

$$\Lambda_N^2 = \frac{\Lambda_{2N}^4}{2\Lambda_{2N}^2 + \mu_{2N}^2} \approx \frac{1}{2}\Lambda_{2N}^2$$

$$\mu_N^2 = \frac{4\mu_{2N}^2\Lambda_{2N}^2 + \mu_{2N}^4}{2\Lambda_{2N}^2 + \mu_{2N}^2} \approx 2\mu_{2N}^2$$

$$\delta_N = \frac{Z_{2N}\Lambda_{2N}^4}{(2\Lambda_{2N}^2 + \mu_{2N}^2)^2} \approx \frac{1}{4}Z_{2N} \tag{3.7}$$

Finally, we relabel the indices so they are sequential, i.e., $a = 2b - 1$ where b runs from $(1, N)$, and $\phi_{2b-1} \equiv \phi_b'$. We then rewrite the Lagrangian:

$$\mathcal{L}_{2N} = Z_N \sum_{b=1}^{N}|\partial\phi_a'|^2 - \Lambda_N^2 \sum_{b=1}^{N}|\phi_b' - \phi_{b+1}'|^2 - \mu_N^2 \sum_{b=1}^{N}|\phi_b'|^2 - \delta_N \sum_{b=1}^{N}|\partial(\phi_b' - \phi_{b+1}')|^2 + \mathcal{O}(\partial^4/\Lambda^2) \tag{3.8}$$

Thus, we have returned to the original theory, $\mathcal{L}_N$, modulo the δ_N term and the higher derivative terms. The presence of the δ_N term, and higher derivatives, reflects the fact that our dedecoration transformation is not exact. These terms are "irrelevant operators," however. The δ_N term has been written in the indicated form because, though it superficially appears to be a relevant $d = 4$ operator, it too is a quartic derivative on the lattice, i.e., (∂^2 in $1 + 3$) $\times$(a nearest neighbor hopping term on the lattice). It effects only the high mass limit of the KK mode spectrum. It will therefore be dropped for consistency with the expansion to order ∂^4/Λ^2.

In eqs.(3.7) we have written the approximate forms of the renormalizations of the parameters in the large Λ_{2N} limit. Note that the μ^2 term is multiplicatively renormalized. This owes to the fact that it is the true scale-breaking term in the theory when the lattice is taken very fine, and Λ terms become derivatives, $i.e.$, as $\mu \to 0$ the theory has a zero-mode. Since it alone breaks the symmetry of scale-invariance, elevating the zero-mode, it is therefore multiplicatively renormalized in free field theory.

We now attempt to define $Q = (D^{-1})^k D^k$ as a product of k decorations followed by k dedecoration transformations. This will, of course, be the identity, and we'll recover the original Lagrangian, but it illustrates the technique for the subsequent nontrivial case involving the BW transformation. We first iterate the decoration transformation of eq.(3.1) k times to obtain a $2^k N$ decorated theory:

$$\mathcal{L}_{2^k N} = Z_{2^k N} \sum_{a=1}^{2^k N} |\partial \phi_a|^2 - \Lambda_{2^k N}^2 \sum_{a=1}^{2^k N} |\phi_a - \phi_{a+1}|^2 - \mu_{2^k N}^2 \sum_{a=1}^{2^k N} |\phi_a|^2 \qquad (3.9)$$

and again assume periodicity, hence $\phi_{2^k N + a} = \phi_a$. The new parameters are chosen so that upon k applications of the dedecoration transformation, we recover the original Lagrangian $\mathcal{L}_N$.

We diagonalize eq.(3.9) with periodic compactification:

$$\phi_a = \frac{1}{\sqrt{2^k N}} \sum_{n=1}^{2^k N} e^{2\pi i n a / 2^k N} \chi_n; \qquad \phi_{a+2^k N} = \phi_a \qquad \text{note: } S = \sum_{a=1}^{2^k N} e^{2\pi i (n-m) a / 2^k N} = 2^k N \delta_{nm} \qquad (3.10)$$

whence:

$$\mathcal{L}_{2^k N}(q^2) = Z_{2^k N} \sum_{n=1}^{2^k N} |\partial \chi_n^{(k)}|^2 - \sum_{n=1}^{2^k N} (4\Lambda_{2^k N}^2 \sin^2(\pi n / 2^k N) + \mu_{2^k N}^2) |\chi_n^{(k)}|^2 \qquad (3.11)$$

Note that we have the zero-mode, corresponding to $n = 0$, which is a singlet (equivalent to choosing $n = 2^k N$, which is outside the first Brillouin zone). Each level within the first Brillouin zone with $n \neq 0$ is degenerate with another level $2^k N - n$, thus forming a doublet. This doubling of energy levels is physical, corresponding to the mode expansion in x^5 in terms of 1, $\sin k_n x^5$ and $\cos k_n x^5$, where the sine and cosine terms are degenerate modes (or equivalently, left–movers and right–movers). With orbifold compactification each level would be a singlet, and what we say presently works as well in the orbifold case.

For KK modes of momentum q^2, we have from the momentum space form of eq.(3.11),

$$\mathcal{L}_{2^k N} = \sum_q \sum_{n=1}^{2^k N} \omega_n^{(k)}(q^2) |\chi_{q,n}^{(k)}|^2 \qquad \omega_n^{(k)}(q^2) = Z_{2^k N} q^2 - 4\Lambda_{2^k N}^2 \sin^2(\pi n / 2^k N) - \mu_{2^k N}^2 \quad (3.12)$$

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The parameters of the k-th order decorated theory may be written in terms of the parameters of the $\mathcal{L}_N$ theory from eqs.(3.7):

$$Z_{2^k N} = 2^{-k} Z_N \qquad \Lambda^2_{2^k N} = 2^k \Lambda^2_N \qquad \mu^2_{2^k N} = 2^{-k} \mu^2_N \tag{3.13}$$

Note that, while we're writing things in momentum space, the actual dedecorations are done in configuration space, $i.e.$, we return to configuration space to integrate out alternating fields. At the ℓth iteration the number of fields remaining is $2^{k-\ell} N$ and we replace the current $\chi^{(\ell)}_{q,n}$ coefficents by new set of $\chi^{(\ell-1)}_{q,n}$, with $2^{k-\ell-1} N$ Fourier coefficients. We then rediagonalize to obtain the new momentum space expression for $\omega^{(\ell)}_n$. This replaces the previous value of $2^{k-\ell} N$ by $2^{k-\ell-1} N$ in the argument of the $\sin^2(\pi n / 2^x N)$ of eq.(3.12).

We thus obtain $(D^{-1})^k D^k$, from the rescalings of eq.(3.7), $i.e.$, $\omega^{(k)}_n \rightarrow \omega^{(0)}_n$, where:

$$\omega^{(0)}_n = 2^{-k} Z_N q^2 - 2^{-k+2} \Lambda^2_N \sin^2(\pi n / N) - 2^{-k} \mu^2_N \tag{3.14}$$

where $2^0 N = N$ now appears in the argument of $\sin^2(\pi n / N)$.

Renormalizing the fields to canonical normalization leads to:

$$\omega^{(0)}_n \rightarrow q^2 - 4\Lambda^2 \sin^2(\pi n / N) - \mu^2 \tag{3.15}$$

where:

$$\Lambda^2 \equiv \Lambda^2_N / Z_N \qquad \mu^2 \equiv \mu^2_N / Z_N \tag{3.16}$$

We have recovered the original theory. Decoration and dedecoration transformations are inverses of one another. Indeed, that is how they were constructed.

As we now see, to define Q, such that the compactification scale $1/R = M = \Lambda\pi/N$ and number of modes N is held fixed, such that we obtain a linear ladder spectrum, we must follow decoration by a transformation that thins degrees of freedom in momentum space.

3.2   Bell-Wilson Transformation

We now consider $Q^k = B^k D^k$ where is B^k a degree-of-freedom thinning transformation that acts upon the decorated theory, $D^k(\mathcal{L}_N) = \mathcal{L}_{2^k N}$ of eq.(3.9). B is the BW transformation, which, contrary to dedecoration, is defined in momentum space. Moreover, the rescaling from $\ell \rightarrow \ell-1$ is now controlled by a new parameter, λ, which is associated with the block-spin wave-function normalization. For a special choice of $\lambda = \lambda^* = 1/\sqrt{2}$ we will scale toward a Gaussian fixed point theory which recovers the linear ladder spectrum.

The parameters of $\mathcal{L}_{2^k N}$, the $X_{2^k N}$, are given in terms of X_N by eq.(3.13). We again diagonalize eq.(3.9) with periodic compactification obtaining eq.(3.11) and adopt the compact notation for $\mathcal{L}_{2^k N}$ of eq.(3.12) in momentum space where the field's $2^k N$ Fourier coefficients are denoted $\chi_{q,n}^{(k)}$.

The Bell-Wilson transformation can be phrased as a mapping from the action $S_{(2^k N)}(\chi^{(k)})$ as a functional of the "old" $2^k N$ variables, $\chi_n^{(k)}$ to a new actiuon, $S_{(2^{k-1}N)}(\chi^{(k-1)})$ in the "new" $2^{k-1} N$ variables, $\chi_n^{(k-1)}$:

$$e^{iS_{(2^{k-1}N)}(\chi^{(k-1)})} = \int D\chi^{(k)} \, T(\chi^{(k-1)}, \chi^{(k)}) e^{iS_{(2^k N)}(\chi^{(k)})} \tag{3.17}$$

We choose a Gaussian block spin redefinition in momentum space, of the form:

$$T(\chi^{(k-1)}, \chi^k) = \exp\left(iK \sum_{n=0}^{2^{k-1}N} |\chi_n^{(k-1)} - \lambda\chi_n^{(k)}|^2 \right) \tag{3.18}$$

Here K is a (large) mass^2 scale parameter, and is formally arbitrarily defined relative to the normalizations of the χ's. K is introduced to engineer a functional delta-function, locking the new $\chi_n^{(k-1)}$ and the old $\chi_n^{(k)}$ together. K should exceed the largest physical scale in the problem, which is the hopping parameter $\Lambda_{2^k N}^2$. Let us therefore define:

$$K = F\Lambda_{2^k N}^2 \qquad F >> 1. \tag{3.19}$$

The key to the BW transformation is that we we integrate out half of the old momentum space modes for a given value of λ. Half of the $\chi_n^{(k)}$ modes, from $n = 1$ to $n = 2^{k-1}N$, are mapped into the new $\chi_n^{(k-1)}$ modes, while the remaining high momentum, or short distance, $n > 2^{k-1}N$ modes are simply integrated out. The $\chi_n^{(k-1)}$ modes are labelled by indices that run from $j = 1$ to $2^{k-1}N$. The fact that only half of the $\chi_n^{(k)}$ momentum modes are mapped into the transformed action means that the $\chi_n^{(k-1)}$ modes are "block spins" in configuration space. It is not hard to see, in configuration space, that one BW transformation yields $2^{k-1}N$ new fields, ϕ_a^{new}, written as blocks spins of the $2^k N$ old fields, ϕ_a^{old} as:

$$\phi_a^{new} = \frac{\lambda}{2\pi} \sum_{b=1}^{2^k N} \frac{|\sin(\pi(a - b/2))|}{|a - b/2|} \phi_b^{old} \tag{3.20}$$

where we have labelled the $2^{k-1}N$ new fields as $1 \lesssim a \lesssim 2^{k-1}N$, and the old ones as $1 \lesssim b \lesssim 2^k N$, and we have absorbed phases into ϕ_b^{old}. λ plays the fundamental role of defining the wave-function normalization of the new spin block spin variables.

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Successive application of the BW transformation k times leads to:

$$e^{iS^{(0)}(\chi^{(0)})} = \int D\chi^{(k)}..D\chi^{(1)} e^{iK \sum_{n=1}^{2^{k-1}N} |\chi^{(k)}-\lambda\chi^{(k-1)}|^2}...e^{iK \sum_{n=1}^{2N} |\chi^{(1)}-\lambda\chi^{(0)}|^2} e^{-\sum_{n=1}^{2^k N} \omega_n^{(2^k N)} |\chi_n^{(k)}|^2}$$

(3.21)

Under the ℓth transformation of the ℓ variables into the $\ell - 1$ variables we find:

$$S_{(2^{(\ell-1)}N)}(\chi^{(\ell-1)}) = K \sum_q \sum_{n=1}^{2^{(\ell-1)}N} \frac{\omega_n^{(\ell)}(q^2)}{K\lambda^2 + \omega_n^{(\ell)}(q^2)} |\chi_{q,n}^{(\ell)}|^2$$

(3.22)

hence:

$$\omega_n^{(\ell-1)} = \frac{\omega_n^{(\ell)}}{(\lambda^2) + \omega_n^{(\ell)}/K}$$

(3.23)

This is a recursion relation which we can easily solve for the general map from $\omega^{(k)}$, corresponding to $\mathcal{L}_{2^k N}$, back to $\omega^{(0)}$, corresponding to $\mathcal{L}_N$. We obtain:

$$S_N(\chi^{(0)}) = \sum_q \sum_{n=1}^{N} \omega_n^{(0)}(q^2) |\chi_{q,n}^{(0)}|^2$$

(3.24)

where:

$$\omega_n^{(0)} = \frac{\omega_n^{(k)}}{(\lambda^{2k}) + c_k \omega_n^{(k)}/K} \qquad c_k = \lambda^{-2k}\left(\lambda^{2+2k} - 1\right)\left(\lambda^2 - 1\right)^{-1}$$

(3.25)

This is the general renormalization group for the kinetic term part of the theory.

For KK modes of momentum q^2, we have for the kth order decorated theory,

$$\omega^{(k)}(q^2) = Z_{2^k N} q^2 - 4\Lambda_{2^k N}^2 \sin^2(\pi n/2^k N) - \mu_{2^k N}^2$$

(3.26)

From the original decoration transformation acting on $\mathcal{L}_N$ we have:

$$Z_{2^k N} \to 2^{-k} Z_N \qquad \Lambda_N^2 = Z_N \Lambda^2 \qquad \Lambda_{2^k N}^2 = 2^k Z_N \Lambda^2 \qquad K = 2^k F Z_N \Lambda^2$$

(3.27)

through the RG results of eq.(3.13). The scale $M = \pi\Lambda_N/N$ is the RG invariant mass of the first KK-mode. We also have:

$$Z_{2^k N}^{-1} \mu_{2^k N}^2 = Z_N^{-1} \mu_N^2 \equiv \mu^2 \qquad \text{hence:} \qquad \mu_{2^k N}^2 = 2^{-k} \mu^2$$

(3.28)

where μ^2 is the RG invariant physical mass scale.

We note that a key difference between the BW transformation and the dedecoration transformation, is that in the ω_n^ℓ at the $k - \ell$th order of iterating the transformation, the the KK-mode mass term involves $\sin^2(\pi n/2^k N)$, and *not* $\sin^2(\pi n/2^\ell N)$! This is a direct consequence of the fact that the BW transformation is defined through eq.(3.18) *in momentum space*, and we do not return to configuration space to integrate out fields!

Finally, for large k and we see that

$$c_k \to \frac{1}{1 - \lambda^2}\lambda^{-2k} \qquad \lambda < 1; \qquad c_k \to 1 \qquad \lambda > 1. \tag{3.29}$$

The final action after k iterations of the BW transformation is summed over N degrees of freedom, and we can therefore write, for $\lambda < 1$:

$$\omega_n^{(0)} = \sum_{n=1}^{N} \frac{2^{-k} Z_N \left(q^2 - 2^{2k+2}\Lambda^2 \sin^2(\pi n/2^k N) - \mu^2\right) |\chi_n^{(0)}|^2}{\lambda^{2k} + \lambda^{-2k}2^{-2k} Z_N \left[(q^2 - 2^{2k+2}\Lambda^2 \sin^2(\pi n/2^k N) - \mu^2)/(1 - \lambda^2)Z_N F\Lambda^2\right]} \tag{3.30}$$

The generality of the BW transformation is contained in the freedom to choose the parameter λ. If $\lambda < 1/\sqrt{2}$ we see that the second term in the denominator dominates in the large k limit and the theory becomes a static (nonpropagating) system,

$$\omega_n^{(0)}(q^2) \to \sum_{n=1}^{N} \lambda^{2k}2^k(1 - \lambda^2)Z_N F\Lambda^2 |\chi_n^{(0)}|^2 \tag{3.31}$$

This is essentially an auxilliary field theory with the effective kinetic term normalization set to zero.

For the special case $\lambda = \lambda^* \equiv 1/\sqrt{2}$ we have $c_k \to (4/3)\lambda^{-2k}$ and the theory approaches a nontrivial fixed point, called the "Gaussian fixed point." We can exercise our freedom to define the original kinetic term normalization constant $Z_N = 1$. For $\lambda = \lambda^*$ we have:

$$\begin{aligned}
\omega_n^{(0)} &= \frac{(q^2 - 2^{-2k+2}\Lambda^2 \sin^2(\pi n/2^k N) - \mu^2)}{1 + 4(q^2 - 2^{-2k+2}\Lambda^2 \sin^2(\pi n/2^k N) - \mu^2)/3F\Lambda^2} \\
&\approx q^2 - 4\Lambda^2 n^2\pi^2/N^2 - \mu^2 + \mathcal{O}(1/F).
\end{aligned} \tag{3.32}$$

which leads to the "linear ladder spectrum" of N KK modes as $k \to \infty$:

$$m_n^2 = 4\Lambda^2(\pi n/N)^2 + \mu^2 \tag{3.33}$$

Therefore, Q defined as a kth order decoration of an N-mode containing deconstructed theory, followed by the kth order Bell-Wilson transformation at the fixed point $\lambda = \lambda^* \equiv 1/\sqrt{2}$ yields a theory with N-modes and the linearized spectrum.

If $\lambda > 1/\sqrt{2}$ we see that the first term in the denominator dominates in the large k limit and the theory scales as:

$$\omega_n^{(0)} \to \lambda^{-2k}2^{-k} Z_N(q^2 - 2^{2k+2}\Lambda^2 \sin^2(\pi n/2^k N) - \mu^2) \tag{3.34}$$

We can define $\lambda^{-2k}2^{-k} Z_N = \eta^{-k}$ where $\eta > 1$:

$$\omega_n^{(0)} \to \eta^{-k}(q^2 - 4\Lambda^2(\pi n/N)^2 - \mu^2) \tag{3.35}$$

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Therefore, $\lambda \gtrsim 1/\sqrt{2}$ defines a fixed line of theories with ladder spectrum. The Gaussian fixed point is presumably stable in the presence of interactions, as in the nonlinear generalization of Bell and Wilson.

4 Discussion and Conclusion

We have considered RG transformations that act in a theory space of N degrees of freedom. This theory space *a priori* knows nothing of extra dimensions and geometry. Nonetheless, a particular RG transformation Q, constructed from a net scale-invariant pair of RG transformations, has a fixed point corresponding to an effective theory of a compactified geometrical extra dimension.

More specifically, we have observed that the combination of a k-fold decoration transformation, followed by a k-fold Bell-Wilson transformation can produce a theory with (i) no rescaling of the number of degrees of freedom, N, and (ii) a fixed point with a linear KK-mode spectrum, when Q is defined with its parameter $\lambda = \lambda^* \equiv 1/\sqrt{2}$. This corresponds to the Gaussian fixed point of the Bell-Wilson transformation.

This has been a preliminary look at the problem of constructing more general Q operators. There remains a great to do. It should be possible to implement this transformation in an interacting theory, in analogy to QCD [9].

Q as we've defined it requires that we exit the microcanonical ensemble of N modes, passing to $2^k N$ modes, as an intermediate step. This is fine for free field theory, but apparently undesirable in the interacting case because the physical interacting theory is bounded in N by the perturbative unitarity. This latter issue however, may be a redherring. It may be possible to make the step outside the microcanonical ensemble if we also make the effective coupling scale, *e.g.*, 2^{-k} times smaller upon decoration, which defines the decoration transformation as holding the high energy coupling constant at the scale $\Lambda_{2^k N}$ fixed.

It would be useful to connect to the differential form of the RG, which is more familiar in QFT perturbation theory. Ultimately, we would prefer a more abstract definition of Q that lives entirely within the microcanonical ensemble of fixed N theories in theory space. A possible route to a more general Q may involve abstracting from the present procedure. It is a simple matter to write the fixed point theory as a Lagrangian of the form:

$$\mathcal{L}_N^* = \sum_{a=1}^{N} |\partial \phi_a|^2 - \Lambda_N^2 \sum_{a=1}^{N} \sum_{b=1}^{N} c_b |\phi_a - \phi_{a+b}|^2 - \mu_N^2 \sum_{a=1}^{N} |\phi_a|^2 \qquad (4.36)$$

where the c_b are chosen to yield the flat spectrum. This involves a simple Fourier transform of the flat spectrum. It would be instructive to frame the action of the kth order Q^k on a generic $\mathcal{L}'_N$, containing arbitrary coefficients, c'_b, as a mapping of $c'_b \rightarrow c_b$, and to observe the approach to the fixed point theory. This may permit an abstraction of Q, which discards the decoration and BW transformations altogether, and a Q that is more practically useful than the detailed procedure we have outline presently. Nonetheless, the detailed procedure we have followed here shows conceptually what is involved in constructing a geometrical RG transformation Q.

We note that a number of the issues regarding locality vs. geometry arise in novel approaches to deconstructing gravity [10]. The RG as a geometrical symmetry may prove to be a useful tool in this arena. These and other related issues are under present consideration.

Acknowledgements

References

[1] C. T. Hill, S. Pokorski and J. Wang, Phys. Rev. D **64**, 105005 (2001);
H. C. Cheng, C. T. Hill, S. Pokorski and J. Wang, Phys. Rev. D **64**, 065007 (2001);
H. C. Cheng, C. T. Hill and J. Wang, Phys. Rev. D **64**, 095003 (2001);
C. T. Hill, Phys. Rev. Lett. **88**, 041601 (2002)

[2] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Rev. Lett. **86**, 4757 (2001);
N. Arkani-Hamed, A. G. Cohen and H. Georgi, JHEP **0207**, 020 (2002);
N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Lett. B **513**, 232 (2001)

[3] T. Bell and K. Wilson, Phys. Rev. **B10**, 3935 (1974)

[4] C. T. Hill, "Fractal theory space: Spacetime of noninteger dimensionality," arXiv:hep-th/0210076, to appear in *Phys. Rev.* **D**. Note: The result quoted for μ^2 in eq.(3.7) is misprinted in the above reference's eq.(2.8), though the correct result for 4-chains is used throughout.

[5] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252 (1941); G. H. Wannier, Rev. Mod. Phys. **17**, 50 (1945).

[6] L. Onsager, Phys. Rev. **65**, 117 (1944).

[7] M. E. Fischer, Phys. Rev. **113**, 969 (1959).

[8] U. J. Wiese, Phys. Lett. B **315**, 417 (1993);
W. Bietenholz and U. J. Wiese, Nucl. Phys. Proc. Suppl. **34**, 516 (1994).

[9] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B **507**, 399 (1997);
T. DeGrand, A. Hasenfratz, P. Hasenfratz and F. Niedermayer, Nucl. Phys. B **454**, 587 (1995);
T. A. DeGrand, A. Hasenfratz, P. Hasenfratz, F. Niedermayer and U. Wiese, Nucl. Phys. Proc. Suppl. **42**, 67 (1995);
P. Hasenfratz and F. Niedermayer, Nucl. Phys. B **414**, 785 (1994).

[10] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, arXiv:hep-th/0210184;
N. Arkani-Hamed and M. D. Schwartz, arXiv:hep-th/0302110;
M. D. Schwartz, arXiv:hep-th/0303114.